Quantum Mechanics on the Hypercube

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Abstract

We construct quantum evolution operators on the space of states, that is represented by the vertices of the n-dimensional unit hypercube. They realize the metaplectic representation of the modular group $SL(2,\mathbb{Z}_{2^n})$. By construction this representation acts in a natural way on the coordinates of the non-commutative 2-torus, $\mathbb{T}^2$ and thus is relevant for noncommutative field theories as well as theories of quantum space-time.

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Recent progress in M-theory indicates that spacetime itself becomes noncommutative at scales where D-branes play an important role [1, 2]. This noncommutativity comes about in a rather natural way because D-branes are charged, gravitational solitons, moving in backgrounds with magnetic flux. This is reminiscent of the Landau problem [3].

On the other hand, quantum mechanics provides a prototype of noncommutativity–but in phase space. In the Landau problem the noncommutativity of the two, real, space coordinates is brought about by the magnetic flux.

In previous work [4, 5, 6], taking advantage of the existence of finite dimensional representations of the Heisenberg-Weyl group, for specific values of Planck's constant, *viz.* $\hbar = 2\pi/N$, we studied quantum spaces [7], i.e. whose coordinates are (finite dimensional) matrices in this group. The object of this exercise was to quantize, exactly, linear cellular automata, using the metaplectic representation of $SL(2,\mathbb{Z}_N)$, where N was any odd integer. These quantum maps have been studied, in particular within the context of quantum chaos [8], Rational Conformal Field Theory [9] and quantum gravity [10].

The case $N = 2^n$ was not amenable to analysis using the tools thus far available, although it is of clear interest for quantum computing [11] and the state space has been widely used in communication engineering. Indeed the principal difficulty resides in resolving ambiguities due to the factors of $1/2$ that abound in the expressions of the metaplectic representation.

In this note we present a prescription that resolves these ambiguities and we construct the metaplectic representation of the symplectic group (linear, canonical, transformations) for the discrete torus, $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ [12]

Our results are useful for studying field theories on noncommutative spaces [13], whose coordinates are finite dimensional matrices [14], as well as fast quantum algorithms [11].

We begin by reviewing some key features of $SL(2,\mathbb{Z}_{2^n})$.

The classical evolution of a linear cellular automaton on the phase space $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ is given by the discrete, canonical, transformation

$$\left(\begin{array}{c} q_{n+1} \\ p_{n+1} \end{array}\right) = \mathsf{A} \cdot \left(\begin{array}{c} q_n \\ p_n \end{array}\right) \tag{1}$$

where $A \in SL(2,\mathbb{Z}_{2^n}) = Sp(1,\mathbb{Z}_{2^n})$. The properties of this group can be analyzed using those of the (abelian) lattice $\mathbb{Z}/\mathbb{Z}_{2^n}$. This lattice contains a subrgoup, consisting of all odd integers mod 2^n;it has 2^{n-1} elements. The main novelty, that distinguishes it from the case of N odd is that the equation

$$x^2 \equiv 1 \bmod 2^n \tag{2}$$

admits two new solutions

$$x_\pm = 2^{n-1} \pm 1 \tag{3}$$

in addition to the old ± 1, so there are *four* units.

The group thus splits into two subgroups: the set consisting of the two non-trivial units and a cyclic subgroup of 2^{n-2} elements, whose primitive element is 3 or 5.

A subgroup of $SL(2,\mathbb{Z}_{2^n})$ of particular interest is the "rotation group", $SO(2,\mathbb{Z}_{2^n})$, comprising matrices of the form

$$\mathsf{A} = \left(\begin{array}{cc} a & b \\ -b & a \end{array}\right) \tag{4}$$

with $a^2+b^2\equiv 1\,\mathrm{mod}\,2^n$. Its structure is relevant for the study of the harmonic oscillator within the framework of Finite Quantum Mechanics and was studied in Ref. [4]

We can find all elements of this group by solving the constraint, $a^2+b^2\equiv 1\,\mathrm{mod}\,2^n$ in terms of an even number t

$$\begin{aligned} a &= 2t(t^2+1)^{-1}\,\mathrm{mod}\,2^n \\ b &= (t^2-1)(t^2+1)^{-1}\,\mathrm{mod}\,2^n \end{aligned} \tag{5}$$

This parametrization, together with that obtained by the exchanges $a\longrightarrow b$, $b\longrightarrow a$ and $a\longrightarrow -b$, $b\longrightarrow a$, gives us all the 2^{n+1} elements.

There is, in particular, a cyclic subgroup of dimension 2^{n-1}, generated by the element corresponding to the value $t=2$ for any n[‡].

After this introduction, we shall propose an explicit expression for the evolution operator, $U(\mathsf{A})$, $\mathsf{A}\in SL(2,\mathbb{Z}_{2^n})$ and we shall prove that it realizes (a) a group representation and (b) a metaplectic representation.

We define

$$\begin{aligned} \omega_n &= e^{2\pi\mathrm{i}/2^n} \\ \widehat{\omega}_n &= e^{2\pi\mathrm{i}/2^{n+1}} \end{aligned} \tag{6}$$

and posit the following expression

$$U\left(\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]\right)_{k,l} = \frac{c_n(\mathsf{A})}{\sqrt{2^n}}\widehat{\omega}_n^{(ak^2-2kl+dl^2)/c} \tag{7}$$

where

$$\mathsf{A}=\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\in SL(2,\mathbb{Z}_{2^n})$$

The constant $c_n(\mathsf{A})$ will be determined by the requirement that U realize a group representation

$$U(\mathsf{A}\cdot\mathsf{B}) = U(\mathsf{A})\cdot U(\mathsf{B})$$

We note here that the expression for $U(\mathsf{A})$ used for N odd depends on ω_n and *not* on $\widehat{\omega}_n$, since $1/2$ existed mod N–whereas it is no longer defined.

The proof of point (a) proceeds as follows:

We set $\mathsf{A}_{ij}=a_{ij}$, $\mathsf{B}_{ij}=b_{ij}$ and $(\mathsf{A}\cdot\mathsf{B})_{ij}=c_{ij}$ and easily find

$$\sum_{m=0}^{2^n-1} U(\mathsf{A})_{k,m}U(\mathsf{B})_{m,l} = \frac{c_n(\mathsf{A}\cdot\mathsf{B})}{c_n(\mathsf{A})c_n(\mathsf{B})}\times U(\mathsf{A}\cdot\mathsf{B})_{k,l}\times\frac{1}{\sqrt{2^n}}\sum_{m=0}^{2^n-1}\widehat{\omega}_n^{\phi} \tag{8}$$

where

$$\phi = \frac{c_{21}}{a_{21}b_{21}}m^2$$

The Gauß sum,

$$\sigma_n(a) = \frac{1}{\sqrt{2^n}}\sum_{m=0}^{2^n-1}\omega_n^{am^2} \tag{9}$$

takes the following values, for $N=2^n$ [15]

[‡]This primitive element is called the Balian-Itzykson oscillator.

- $$\sigma_n(1) = 1 + \mathrm{i}$$

  for $n > 2$.

- If a is an odd integer,

  $$\sigma_n(a) = (-2^n|a)\varepsilon(a)(1+\mathrm{i})$$

  where we have used the symbol

  $$\varepsilon(a) = \begin{cases} 1 & a \equiv 1 \bmod 4 \\ \mathrm{i} & a \equiv 3 \bmod 4 \end{cases}$$

and the Jacobi symbol[§]

$$(a|b) = \begin{cases} 1 & a \equiv x^2 \bmod b \\ -1 & a \not\equiv x^2 \bmod b \end{cases}$$

It is also possible to show that

$$\sum_{m=0}^{2^n-1} \widehat{\omega}_n^{m^2 a} = \sqrt{2^{n-1}}\sigma_{n+1}(a)$$

More properties of this sum may be found in Lang's text.

Choosing the constant $c_n(\mathsf{A})$ in eq. (7)

$$c_n(\mathsf{A}) = -\frac{\mathrm{i}\sigma_{n+1}(c)}{\sqrt{2}}$$

it follows that the evolution operator

$$U(A)_{k,l} = \frac{-\mathrm{i}\sigma_{n+1}(c)}{\sqrt{2^{n+1}}} \widehat{\omega}_n^{(ak^2-2kl+dl^2)/c} \tag{10}$$

realizes an $N = 2^n$-dimensional representation, which provides the *exact* quantization of all linear cellular automata, $\mathsf{A} \in SL(2, \mathbb{Z}_{2^n})$.

In the above derivation we assumed that the element $c = \mathsf{A}_{21}$ is odd, in order that its inverse in the exponent be defined. When $c = \mathsf{A}_{21}$ is even, then both $a = \mathsf{A}_{11}$ and $d = \mathsf{A}_{22}$ must be odd (since $ad - bc \equiv 1 \bmod 2^n$). In this case we *define* $U(\mathsf{A})$ as

$$U(\mathsf{A}) = U(\mathsf{A} \cdot \mathsf{S}) \cdot U^{-1}(\mathsf{S}) \tag{11}$$

where

$$\mathsf{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

This definition, for $c = \mathsf{A}_{21}$ even, is chosen in order to cover all the cases of the products of any two matrices, $\mathsf{A}, \mathsf{B} \in SL(2, \mathbb{Z}_{2^n})$ and guarantees that U indeed defines a representation.

[§]The notation $\left(\frac{a}{b}\right)$ is also used.

One nontrivial check of this is provided by the result

$$U(\mathsf{A})^{N/2} = U(\mathsf{A}^{N/2}) = U(I_{2\times 2}) = I_{2^n\times 2^n} \tag{12}$$

for $\mathsf{A} \in SO(2,\mathbb{Z}_{2^n})$, which shows that the space of eigenstates splits into two copies, indexed by a parity quantum number. This is to be contrasted to the result obtained for $N = 4k \pm 1$ prime, where $U(\mathsf{A})^{4k} = I_{N\times N}$. Eq. (12) follows from the theorem:

Theorem 1 *For $N = 2^n$, $n > 2$ and $\mathsf{A} \in SO(2,\mathbb{Z}_N)$*

$$\mathsf{A}^{N/2} \equiv I_{2\times 2} \bmod N$$

Proof: This is established by induction. Setting $\mathsf{a}_1 = a, \mathsf{b}_1 = b$, we find that

$$\begin{aligned} \mathsf{a}_{k+1} &= \mathsf{a}_k a - \mathsf{b}_k b \\ \mathsf{b}_{k+1} &= \mathsf{a}_k b + \mathsf{b}_k a \end{aligned} \tag{13}$$

For $n = 3$, we find that

$$\mathsf{a}_4 = 8a^4 - 8a^2 + 1 \equiv 1 \bmod 8 \tag{14}$$

which proves the first step. Assuming the theorem holds for some $n > 3$, we will show it holds for $n+1$, *viz.*

$$\mathsf{a}_{2^{n-1}} \equiv 1 \bmod 2^n \Rightarrow \mathsf{a}_{2^n} \equiv 1 \bmod 2^{n+1} \tag{15}$$

Indeed,

$$\begin{aligned} \mathsf{a}_{2^n} = \mathsf{a}^2_{2^{n-1}} - \mathsf{b}^2_{2^{n-1}} = 2\mathsf{a}^2_{2^{n-1}} - 1; \\ \mathsf{a}_{2^{n-1}} = 1 + 2^n r \Rightarrow 2\mathsf{a}^2_{2^{n-1}} - 1 = 2^{n+1}(2^n r^2 + 2r) + 1 \equiv 1 \bmod 2^{n+1} \end{aligned} \tag{16}$$

We note that a_l, as a function of a,is a Chebyshev polynomial of degree l. The theorem just proven is equivalent to the amusing property that the value of the order $l = 2^n > 4$ polynomial, for any, even, value of its argument, is equal to $1 \bmod 2^{n+1}$.

The last step of our construction is to prove that the evolution operator $U(\mathsf{A})$, constructed above, has the *metaplectic* property, thus providing the exact quantization of the action of linear, classical, maps $\mathsf{A} \in SL(2,\mathbb{Z}_{2^n})$. We start by recalling the definition of the generators of the Heisenberg–Weyl group

$$J_{r,s} = \widehat{\omega}_n^{r\cdot s} P^r \cdot Q^s, \quad r,s = 0,\ldots,2^n-1 \tag{17}$$

where

$$P_{k,l} = \delta_{k-1,l}, \quad k,l = 0,\ldots,2^n-1$$

$$Q_{k,l} = \omega_n^k \delta_{k,l}, \quad k,l = 0,\ldots,2^n-1$$

The metaplectic property, defined by the relation

$$U(\mathsf{A}) J_{r,s} U^{-1}(\mathsf{A}) = J_{(r,s)\mathsf{A}} \tag{18}$$

guarantees that $U(\mathsf{A})$ quantizes the action of A on the classical phase space.

It may be easily checked by a direct computation, that the operator, given by eq. (10) indeed satisfies this definition.

Furthermore, since the representation of the Heisenberg-Weyl group is irreducible and the representation of $SL(2,\mathbb{Z}_{2^n})$ just constructed satisfies the metaplectic property, we deduce that this representation is irreducible.

This paper ends the cycle [4, 5]. All the necessary tools, for dealing with the evolution operator of any quantum system within the framework of finite quantum mechanics, are now operative. Indeed, for any value of N, using the Chinese remainder theorem, for the prime factors $2^n \times N'$, where N' is odd, it is possible to construct the metaplectic representation of $SL(2,\mathbb{Z}_N) = SL(2,\mathbb{Z}_{2^n}) \times SL(2,\mathbb{Z}_{N'})$ by tensoring the representations of the corresponding factor groups [5]. They open several directions that merit further investigation:

In recent work on noncommutative gauge theory [2, 13, 14] the coordinates satisfy non-trivial commutation relations, $[X_\mu, X_\nu] = C_{\mu\nu}$. Our work indicates that it may also be of interest to consider, instead, position operators that take values in the group and not the algebra, that is impose $X_\mu X_\nu = \omega X_\nu X_\mu$, where ω is a root of unity (Manin's quantum plane, cf. [7]). In this, more general situation, one might expect to recover at will classical/quantum dynamics and/or (non)commutative geometric effetcs, depending on how the scaling limit is taken (i.e. what is held fixed as $N \to \infty$). A simple, but non-trivial, example is given by the Azbel-Hofstadter Hamiltonian [6] that describes a charged particle moving on the plane in the presence of a uniform transverse magnetic field and a periodic scalar potential, whose spectrum, for finite, rational flux/plaquette, is generated by deformations of the $SU(2)$ algebra, which, in the limit of infinitesimal flux, becomes a classical $SU(2)$ algebra for the quantum dynamics. Using elements of the Heisenberg-Weyl group as coordinates it would be interesting to calculate, in perturbation theory, properties of noncommutative Yang-Mills theories, because the finite dimension of the Heisenberg-Weyl group provides, at the same time, an UV and IR cutoff [2, 13].

Another interesting application concerns quantum algorithms, which, to date [11], are based mainly on the Fourier transform, which is related to the evolution operator of the harmonic oscillator. We have considered arbitrary quantum maps, for which the corresponding algorithms remain to be constructed.

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